Exercise 8.2.2

Show that the Hermite ODE, Table 7.1, may be put into self-adjoint form by multiplying by e^{-x^2} and that this gives $w(x) = e^{-x^2}$ as the appropriate weighting function.

Solution

From Table 7.1 on page 345, Hermite's equation is

$$y'' - 2xy' + 2\alpha y = 0.$$

At the moment it is not self-adjoint because

$$\frac{d}{dx}(1) \neq -2x.$$

However, if both sides of Hermite's equation are multiplied by the weight function $w(x) = e^{-x^2}$, then it becomes self-adjoint

$$e^{-x^2}y'' - 2xe^{-x^2}y' + 2\alpha e^{-x^2}y = 0 \tag{1}$$

because

$$\frac{d}{dx}(e^{-x^2}) = -2xe^{-x^2}.$$

Equation (1) can therefore be written as

$$e^{-x^2}y'' + \frac{d}{dx}(e^{-x^2})y' + 2\alpha e^{-x^2}y = 0,$$

or

$$\frac{d}{dx}(e^{-x^2}y') + 2\alpha e^{-x^2}y = 0.$$